# A control theoretical approach to the multivariable spectral factorization problem.

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August 2009

## Abstract

It is shown how with analogy to analogue control theory, that the problem of polynomial-matrix spectral factorization can be solved using negative feedback. This recursive method is particularly simple to implement as compared with other approaches. The algorithm in its basic form requires no matrix inversions or special matrix decompositions.


## 1.Introduction

The multivariable spectral factorization problem can be mathematically defined as follows:

Given a finite matrix Laurent-series $\boldsymbol{L}(z)$ in the z-domain of degree n,

$$\boldsymbol{L}(z) = \boldsymbol{D}_{-n}z^{-n} + \boldsymbol{D}_{-(n-1)}z^{-(n-1)} + \dots + \boldsymbol{D}_{-1}z^{-1} + \boldsymbol{D}_0 + \boldsymbol{D}_1 z^1 + \boldsymbol{D}_2 z^2 + \dots + \boldsymbol{D}_n z^n \quad (1)$$

where $\boldsymbol{D}_{-i} = \boldsymbol{D}_i^T, i = 1, 2, \dots n$ are real valued matrices of dimension m-square and $z^{-1}$ represents the backward-shift operator.

Now (1) can be factorized into two polynomial matrices, one of which is analytic within $|z| = 1$, and a second polynomial matrix which is analytic outside $|z| = 1$. This results in a left spectral factorization

$$\boldsymbol{L}(z) = \boldsymbol{A}(z^{-1})\boldsymbol{A}^T(z) \quad (2)$$

where the left spectral factor polynomial-matrix $\boldsymbol{A}(z^{-1}) = \boldsymbol{A}_0 + \boldsymbol{A}_1 z^{-1} + \boldsymbol{A}_2 z^{-2} + \dots \boldsymbol{A}_n z^{-n}$ is full rank and strictly Hurwitz. ie the zeros of $|\boldsymbol{A}(z^{-1})| = 0$ all lie within the unit-circle in the z-plane. The coefficient matrices of $\boldsymbol{A}(z^{-1})$ are all m-square and although the product (2) is unique, the spectral factor itself is not since any m-square matrix $\boldsymbol{E}$ can be removed from the spectral factor giving $\boldsymbol{A}(z^{-1}) = \boldsymbol{A}'(z^{-1})\boldsymbol{E}$ . It is thus often convenient to normalise the first coefficient matrix of the spectral factor polynomial matrix such that $\boldsymbol{A}(0) = \boldsymbol{I}$ giving a unique spectral factor

$$\boldsymbol{L}(z) = \bar{\boldsymbol{A}}(z^{-1})\boldsymbol{R}_L\bar{\boldsymbol{A}}^T(z) \quad (3)$$

where $\boldsymbol{R}_L$ is a constant positive-definite matrix. We can likewise also define a Hurwitz right spectral factor polynomial-matrix $\boldsymbol{B}(z^{-1})$ with $\boldsymbol{B}(0) = \boldsymbol{I}$ such that

$$\boldsymbol{L}(z) = \bar{\boldsymbol{B}}^T(z)\boldsymbol{R}_R\bar{\boldsymbol{B}}(z^{-1}) \quad (4)$$

Unless otherwise specified we will work with the ordinary and not unique spectral-factor definitions.

The idea of spectral-factorization problem was first used by Wiener [1] (continuous-time) and Kolmogorov [2] for the discrete-time scalar case. This was as applied to optimal-filtering and the problem has been solved by many authors including Youla [3],Anderson[4],Callier[5] and Kucera[6]. The spectral-factorization problem also applies to optimal $H_2$ control [7] and $H_\infty$ control [8]. The usual approach to solving the spectral factorization problem is to use Cholesky factorization of matrices eg Kucera [6] or the Baur or Schur methods [9] which are compared in [10] . A Newton-Raphson method is shown in [11] where spectral factorization is used to stabilise discrete polynomials whose roots lie outside the unit-circle. It is known that spectral factorization is the polynomial-equation equivalent of solving a matrix Riccati equation[12].

**3.Feedback Control Multivariable Spectral Factorization**

We consider first the solution to the left polynomial matrix spectral factorization problem. Define a matrix $\boldsymbol{V}$ of dimension $m(n+1)$ x $m$ composed of the elements of the causal part of a known matrix-Laurent series $\boldsymbol{V} = [\boldsymbol{D}_0, \boldsymbol{D}_{-1} ... \boldsymbol{D}_{-n}]^T$ . *only the terms in negative power of z are considered*. If the matrices with positive powers of z are used then the algorithms will not converge. The positive terms are found from

$$\boldsymbol{D}_i = \sum_{j=0}^{n-i} \boldsymbol{A}_j \boldsymbol{A}_{i+j}^T, i = 0,1...n \tag{5}$$

and the negative powers follow from $\boldsymbol{D}_{-i} = \boldsymbol{D}_i^T$ . Consider also another matrix $\boldsymbol{W}$ also of dimension $m(n+1)$ x $m$ initialised to zero or some small diagonal value. Then a multivariable control-loop can be defined as

$$\boldsymbol{W}_{k+1} = \boldsymbol{W}_k + \eta \boldsymbol{\varepsilon}_k \tag{6}$$

where an error matrix is given by

$$\boldsymbol{\varepsilon}_k = \boldsymbol{V} - \boldsymbol{W}_k * \boldsymbol{W}_{-k} \tag{7}$$

and $\eta$ is a small gain (or step-size) usually less than unity. In (7) above, * denotes the convolution of a $\boldsymbol{W}_k$ "impulse response" matrix with its un-causal counterpart. This is akin to just creating another Laurent series based on estimated instead of known values. That is $\hat{\boldsymbol{A}}(z^{-1})\hat{\boldsymbol{A}}^T(z)$ and using only the terms in powers of negative z, resulting in a solution matrix of equal dimension after the convolution as compared to before. This is best described in the following algorithm:

***Algorithm 3.1 Left Polynomial-Matrix Spectral Factorization***

*Step 1*

Initialise an $m(n+1)$ x $m$ matrix to some small value, say $W_0$ . Select a step-size of initially $\eta = 0.01$ and define the Laurent matrix from known matrices that make up the Laurent series: $V = [D_0, D_{-1} ... D_{-n}]^T$ .

*Step 2*

Update an $m(n+1)$ x $m$ error matrix $\varepsilon_k = V - C_k$ where the $C_k = [c_0^T, c_1^T ... c_n^T]^T$ matrix is composed of transposed sub-matrices of $W_k = [w_0, w_1, ... w_n]^T$ given by the convolution $c_i = \sum_{j=0}^{n-i} w_j w_{i+j}^T, i = 0,1...n$ . The sub-matrices $w_i$ and $c_i$ ,i=1,2...n are all m-square.

*Step 3*

Update the $W_k$ matrix according to $W_{k+1} = W_k + \eta \varepsilon_k$

*Step 4*

For some applications in recursive estimation problems we can return to Step 2 and continue forever. However, as a one-shot algorithm for estimating the spectral factor we may require a stopping criterion. In which case we check if the norm $\|\varepsilon_k\|_2 < \delta$ where $\delta$ is some small number depending on the required accuracy (say $10^{-4}$ ) and if this is the case we stop, else return to step 2. The left spectral factor coefficient matrices are composed of the coefficient matrices of the $W_k$ matrix giving $W_k \rightarrow [A_0, A_1, ... A_n]^T$ .

*A simple explanation.*

Some further explanation is required as to how such an unconventional approach can be made to work. Examining Figure 1 we see that the above algorithm consists of a multivariable control-system with set-point $V$ , the known Laurent series (causal part), and a convolution in the feedback path. Consider the simplest case when $n = 0$ , the Laurent series is then a constant positive-definite matrix $D_0$ and the loop gain must via feedback drive the error matrix close to zero giving from (7) $V - w_0.w_0^T = 0$ or $D_0 - w_0.w_0^T = 0$ or $w_0 \rightarrow A_0$ . In fact in the scalar polynomial case when $n = 0$ ,the control-loop has a squared-term in the feedback path which must result in a square-root at the output. So the multivariable control-system drives the output to the “square-root” of $A(z^{-1})A^T(z)$ . Furthermore if the Laurent-series is time-variant then the set-point matrix becomes a time-variant $V_k$ , and the loop will track the spectral factor as it changes with time. Note also that the algorithm requires no divisions, eigenvalue or special matrix decompositions or matrix inversions, but is defined entirely by multiplication and addition/subtraction. Unfortunately there is no exact measure of knowing the step-size at present, but generally a larger value will give faster convergence and smaller value will give slower yet smoother results. By increasing the step-size the bandwidth (unity-gain crossover frequency) of the multivariable integrator increases giving rise to a faster response-time.

*Algorithm 3.2 Right Polynomial-Matrix Spectral Factorization*

We use $[\boldsymbol{A}(z^{-1})\boldsymbol{A}^T(z)]^T$ and use this result to find the right spectral factor.

Follow the previous left spectral factorization algorithm except at Step 1 we transpose the sub-matrices of $\boldsymbol{V}$ giving $\boldsymbol{V} = [\boldsymbol{D}_0^T, \boldsymbol{D}_{-1}^T ... \boldsymbol{D}_{-n}^T]^T$ and the resulting converged polynomial matrix spectral-factor must also be transposed giving $\boldsymbol{W}_k^T \rightarrow [\boldsymbol{B}_0, \boldsymbol{B}_1, ...\boldsymbol{B}_n]^T$ .

**4. Illustrative examples.**

Consider an example given by Kucera [6].

Example 1. Find the left spectral factor of the following matrix Laurent series:

$$\boldsymbol{L}(z) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} z^{1} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} z^{2}$$

Algorithm 3.1 herein gives a solution of

$$\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 0.6852 & 0.0326 \\ 0.0326 & 2.9033 \end{bmatrix} + \begin{bmatrix} -0.1794 & 0.1631 \\ 0.0326 & -0.6852 \end{bmatrix} z^{-1} + \begin{bmatrix} -0.03263 & 0.6852 \\ 0 & 0 \end{bmatrix} z^{-2}$$

Which converged in 30 steps with a step-size of 0.23. From [6] we get from Cholesky factorization

$$\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 0.6860 & 0 \\ 0.1715 & 2.9155 \end{bmatrix} + \begin{bmatrix} -0.1715 & 0.1715 \\ 0 & -0.6860 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & 0.6860 \\ 0 & 0 \end{bmatrix} z^{-2}$$

which converged in 7 steps. The Cholesky method converges in less steps but is far more complex to implement. Both used an error of 0.0001 as a stopping criterion. To back-calculate the matrix Laurent series from these values will give near identical results.

Example 2. Consider the following polynomial-matrix $\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} z^{-1} + \begin{bmatrix} 3 & 0.5 \\ 0.5 & 3 \end{bmatrix} z^{-2}$ which has 4 zeros at -1.5 ± 1.11803j and -0.5 ± 1.5j. The polynomial-matrix is clearly non-minimum phase. Forming the Laurent series and applying Algorithm 3.1 returns the minimum-phase polynomial matrix $\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 3 & 0.5 \\ 0.5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} z^{-2}$ , the zeros of which are now at -0.2 ± 0.6j and -0.43 ± 0.32j. This is in fact the reciprocal polynomial-matrix and the method can be used to stabilise polynomial-matrices by reflecting back within the unit circle any non-minimum phase zeros[11].

## 5.Conclusions

A new method of polynomial-matrix spectral factorization has been demonstrated using multivariable feedback control. The technique is new in that a matrix convolution appears in the feedback path of the control-system. This gives a closed-loop system acting as a matrix square-root algorithm. The technique can be used for left or right matrix spectral factorization and has applications in many areas of control and signal-processing.

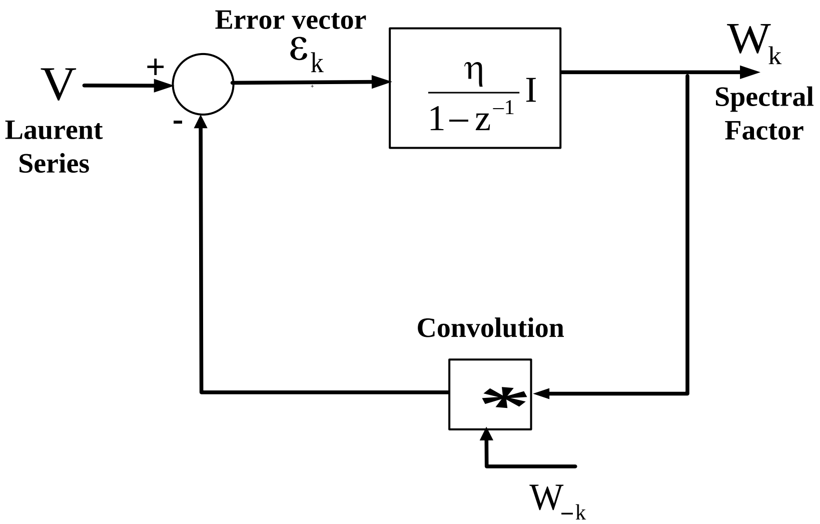


**Figure 1. Polynomial-Matrix Spectral factorization via feedback.**

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